The Imperfection Sensitivity of Isotropic Two-Dimensional Elastic Lattices

Digby D. Symons

Norman A. Fleck

Engineering Department,
Cambridge University,
Trumpington Street,
Cambridge CB2 1PZ, UK

The imperfection sensitivity of the effective elastic properties is numerically explored for three planar isotropic lattices: fully triangulated, the Kagome grid, and the hexagonal honeycomb. Each lattice comprises rigid-jointed, elastic Euler–Bernoulli beams, which can both stretch and bend. The imperfections are in the form of missing bars, misplaced nodes, and wavy cell walls. Their effect on the macroscopic bulk and shear moduli is numerically investigated by considering a unit cell containing randomly distributed imperfections, and with periodic boundary conditions imposed. The triangulated and Kagome lattices have sufficiently high nodal connectivities that they are stiff, stretching dominated structures in their perfect state. In contrast, the perfect hexagonal honeycomb, with a low nodal connectivity of 3, is stretching dominated under pure hydrostatic loading but is bending dominated when the loading involves a deviatoric component. The high connectivity of the triangulated lattice confers imperfection insensitivity: Its stiffness is relatively insensitive to missing bars or to dispersed nodal positions. In contrast, the moduli of the Kagome lattice are degraded by these imperfections. The bulk modulus of the hexagonal lattice is extremely sensitive to imperfections, whereas the shear modulus is almost unaffected. At any given value of relative density and level of imperfection (in the form of missing bars or dispersed nodal positions), the Kagome lattice has a stiffness intermediate between that of the triangulated lattice and the hexagonal honeycomb. It is argued that the imperfections within the Kagome lattice switch the deformation mode from stretching to a combination of stretching and bending. Cell-wall waviness degrades the moduli of all three lattices where the behavior of the perfect structure is stretching dominated. Since the shear response of the perfect hexagonal honeycomb is by bar bending, the introduction of bar waviness has a negligible effect on the effective shear modulus. [DOI: 10.1115/1.2913044]

1 Introduction

There is a current interest in the mechanical properties of two-dimensional and three-dimensional lattice materials. For example, 2D lattices exist as an array of ceramic prismatic tubes in catalytic converters for automotive use. 2D lattices are also used in woven composites, such as triaxially woven carbon fiber epoxy laminates. Natural 2D lattices include the wax honeycomb of the honey bee, wood, and coral. In contrast, bone and sponges form 3D lattices, and 3D woven composites are under development for sandwich cores. In practical applications, these structures are loaded in the elastic regime and it is important to understand the relationship between their microstructure and mechanical properties. We consider in this paper the fundamental problem of the elastic properties of 2D imperfect but isotropic lattices.

It is now well established that the effective properties of lattices are dependent on the degree of nodal connectivity, but no systematic studies have been performed on the imperfection sensitivity of competing lattices of widely varying connectivity. Practical 2D and 3D lattice materials contain imperfections in the form of irregular cells, wavy bars, and possibly missing bars. It is of broad engineering significance to determine imperfection sensitivity of properties: If an imperfection causes a significant drop in a useful property then it may be worthwhile to put in serious effort to manufacture the lattice in as perfect a state as possible. This paper describes an investigation into the imperfection sensitivity of the elastic moduli of three planar *isotropic* lattices: the fully triangulated lattice with a nodal connectivity of $Z=6$, the Kagome lattice with a nodal connectivity of $Z=4$, and the hexagonal honeycomb with a nodal connectivity of $Z=3$, see Fig. 1. All three perfect lattices comprise uniform cell walls (bars) of length l and thickness t.

Contributed by the Applied Mechanics Division of ASME for publication in the Journal of Applied Mechanics. Manuscript received September 28, 2007; final manuscript received October 11, 2007; published online July 17, 2008. Review conducted by Robert M. McMeeking.

1.1 Recent Work on the Mechanical Properties of Planar Isotropic Lattices.

Deshpande et al. [1] have shown that whether a lattice is bending or stretching dominated may be informed by a consideration of its nodal connectivity Z, that is, the number of bars attached to each node. The approach is based on Maxwell's [2] equation for the rigidity of pin-jointed structures. If a pin-jointed structure contains a collapse mechanism under a particular loading, then the equivalent rigid-jointed structure will be flexible (bending dominated) under the same loading. Otherwise, both the pin-jointed and equivalent rigid-jointed structures will be stiff (stretching dominated). The necessary (but not sufficient) condition for rigidity of a planar lattice is $Z=4$. The Kagome lattice only just achieves this condition, and we shall show in this study that this has a major impact on its imperfection sensitivity.

The hexagonal honeycomb architecture (Fig. 1(*c*)) is ubiquitous in both natural and man-made materials. Although stiff under equibiaxial (i.e., hydrostatic) loading, it is flexible under deviatoric loading due to its low nodal connectivity. Hydrostatic loading is resisted by stretching of the cell walls, whereas deviatoric loading is resisted by bending of the cell walls. In contrast, the fully triangulated lattice (Fig. 1(*a*)) is stretching dominated under all loading states and is thereby stiff under both hydrostatic and shear loading.

Hyun and Torquato [3] have shown that the Kagome lattice attains the upper Hashin–Shtrikman bound for the stiffness of a two-phase isotropic composite, where one phase is empty space.

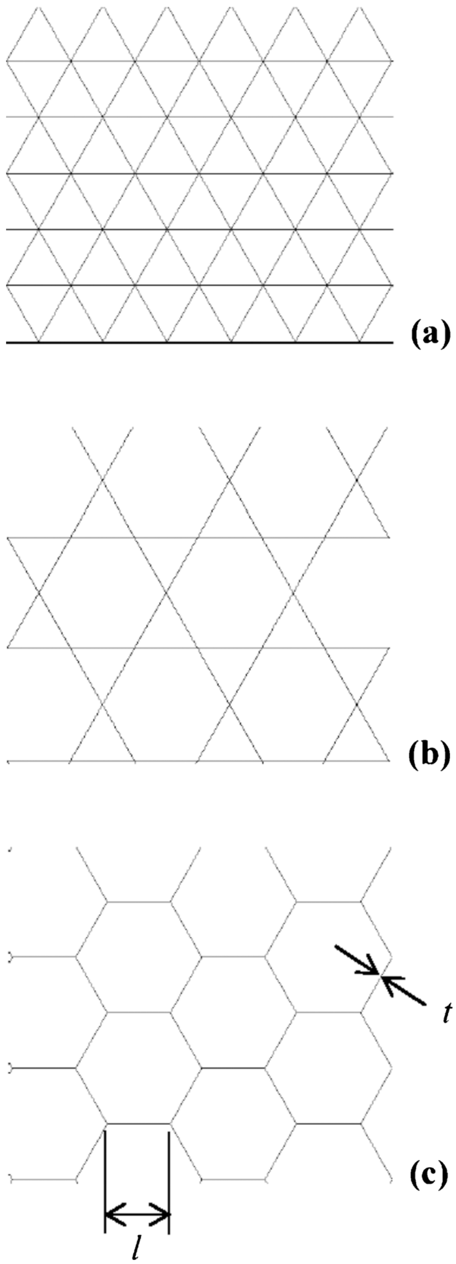

Fig. 1 Perfect geometries of three planar grids: (*a*) triangular, (*b*) Kagome, and (*c*) Hexagonal

In fact, the moduli of the Kagome lattice are identical to those of a triangulated grid of the same relative density $\bar{\rho}$. The high stiffness of the rigid-jointed Kagome lattice is consistent with the collapse response of the infinite pin-jointed version. Hutchinson and Fleck [4] have shown that the infinite pin-jointed Kagome lattice has an infinite number of internal periodic collapse mechanisms, yet none of these mechanisms produces macroscopic strain. Consequently, the Kagome lattice is stretching dominated under both hydrostatic and deviatoric loading.

Symons et al. [5] have explored the morphing potential of the Kagome lattice. They explored the static and kinematic determinacy of the finite, 2D pin-jointed Kagome truss and the finite, double layer grid with Kagome faces. They showed that the addition of patch bars to the periphery of the finite trusses converts them to forms which are statically and kinematically determinate. Thus, on external loading, the structure is stiff. However, if one of the bars is replaced by an axial actuator and extended, the remaining structure behaves as a mechanism with a single degree of freedom. Thus, the structure has a high morphing capability. In order to find application, however, additional studies are needed on the stiffness of the Kagome lattice and its imperfection sensitivity.

The low nodal connectivity of the rigid-jointed Kagome lattice also conveys to it a high fracture toughness. Fleck and Qiu [6] have shown that the lattice deforms by bar stretching remote from the crack tip and by a combination of bar bending and bar stretching within a characteristic elastic deformation zone near the crack tip. This elastic zone reduces the stress concentration at the crack tip in the Kagome lattice and the macroscopic toughness thereby exceeds that of the triangulated and hexagonal lattices.

1.2 Imperfection Sensitivity. Structures contain imperfections, either due to manufacturing or damage. The main aim of this study is to compare the imperfection sensitivity of elastic properties for the triangular lattice, the Kagome lattice, and the hexagonal honeycomb. It is anticipated that the large variation in nodal connectivity from one microstructure to the next will lead to marked differences in imperfection sensitivity. Partial information on the imperfection sensitivity of particular lattices can be gleaned from the existing literature, but no systematic comparisons have been reported to the authors' knowledge.

The sensitivity to node misplacement has been investigated by a number of authors for both 2D hexagonal honeycombs and 3D foams. Silva et al. [7] compared the elastic properties of 2D random Voronoi honeycombs to those of perfect hexagonal honeycombs. The Voronoi honeycomb is an arrangement of irregular hexagons and thereby has a connectivity $Z=3$. They found that the Voronoi honeycomb has a macroscopic shear modulus and Young's modulus slightly higher than that of a perfect hexagonal honeycomb (by 11% and 6%, respectively, for a relative density of $\bar{\rho}=0.15$). They concluded that the random (Voronoi) honeycomb has similar elastic properties to that of the perfect honeycomb. This conclusion is accurate for deviatoric loading but does not hold for hydrostatic loading; Zhu et al. [8] also observed an increase in shear and Young's modulus for increasingly imperfect 2D Voronoi honeycombs but showed that the bulk modulus decreased significantly. This increase in Young's modulus with increasing node misplacement in 2D hexagonal honeycombs is also observed for 3D open cell foams, see, for example, Van der Burg et al. [9]. Both structures are bending dominated in their perfect forms. In contrast, Grenestedt and Tanaka [10] show that perturbing the nodal positions of a 3D closed cell foam (a stretching-dominated structure in its perfect form) leads to a decrease in both bulk and shear moduli.

Chen et al. [11] examined the stiffness (and strength) of hexagonal honeycombs with imperfections in the form of rigid inclusions, holes, and missing cell walls. They found that the honeycomb was insensitive to the presence of rigid inclusions but that the presence of holes or missing cell walls causes a substantial knockdown in bulk modulus due to the induced cell-wall bending. Gan et al. [12] observed similar imperfection sensitivity in 3D open cell foams. Both Young's modulus and bulk modulus were reduced by the presence of broken cell edges, with the bulk modulus showing the greatest knockdown. However, stretching-dominated structures do not show the same sensitivity. Wallach and Gibson [13] found that Young's modulus of a fully triangulated 3D truss material is relatively insensitive to the removal of bars. For example, the removal of 10% of struts from the 3D truss decreases Young's modulus by only 17%.

The effect of cell-wall waviness in open and closed cell 3D foams has been investigated by Grenestedt [14]. A significant knockdown in bulk modulus of the open cell foam was observed for increasing bar waviness, with Young's modulus and shear modulus showing lesser sensitivities. The knockdown observed for closed cell foam was less significant and this was attributed to the dominance of stretching behavior in all deformation modes of this 3D structure, even in the presence of cell-wall waviness. Si-

Table 1 Relevant mechanical properties of the three perfect isotropic lattices

Topology	Relative density $\bar{\rho}$	Bulk modulus K	Shear modulus G	Poisson ratio ν
Triangular honeycomb	$2\sqrt{3}\frac{t}{l}$	$\frac{1}{4}\bar{\rho}E_s$	$\frac{1}{8}\bar{\rho}E_s$	1/3
Hexagonal honeycomb	$\frac{2}{\sqrt{3}}\frac{t}{l}$	$\frac{1}{4}\bar{\rho}E_s$	$\frac{3}{8}\bar{\rho}^3E_s$	1
Kagome lattice	$\sqrt{3}\frac{t}{l}$	$\frac{1}{4}\bar{\rho}E_s$	$\frac{1}{8}\bar{\rho}E_s$	1/3

mone and Gibson [15] also showed a knockdown in Young's modulus due to the cell-wall waviness of both 2D hexagonal honeycombs and closed cell 3D foams; they showed that the reduction is small unless the imperfection is very large.

The interaction of different types of imperfection in cellular materials may be of concern. However, Grenestedt [16] and Li et al. [17] examined combined imperfections in closed cell foams and Voroni honeycombs, respectively, and showed very little interaction between types of imperfection. For small levels of imperfection, the effects are additive. The imperfection sensitivity of Kagome type structures has received limited attention. Symons et al. [18] performed a number of morphing experiments on 3D Kagome lattices and showed that the presence of geometric imperfections significantly reduced the stiffness of the lattice against actuation. However, the imperfection sensitivity of in-plane stiffness under external loads was not explored.

1.3 Outline of the Study. The structure of this paper is as follows. First, the in-plane properties of the perfect triangulated, Kagome, and hexagonal lattices are reviewed. The numerical procedure for predicting the moduli of imperfect lattices is then described. The imperfection sensitivity of each of the three lattices is detailed for three types of imperfection: missing bars, a dispersion of nodes, and bar waviness. This paper concludes with a discussion of the relationship between imperfection sensitivity and nodal connectivity.

2 Review of the In-Plane Properties of Isotropic Lattice Materials

The relative density $\bar{\rho}$ of the three planar isotropic lattices are listed in Table 1 in terms of the cell-wall thickness t and length l. The table also gives the in-plane effective elastic properties in terms of $\bar{\rho}$ and Young's modulus E_s of the parent material, see Gibson and Ashby [19], Christensen [20], Fleck and Qiu [6], and Srikantha Phani et al. [21]. Define the bulk modulus K as the ratio of the mean applied biaxial stress to the in-plane hydrostatic strain. The in-plane shear modulus G has its usual definition.

Note that the elastic moduli of the triangular and Kagome lattices scale linearly with $\bar{\rho}$ in identical fashions. For both microstructures, the behavior is stretching dominated; we neglect here the very small additional contribution to stiffness associated with bending and shear deformation of the struts. The bulk modulus K of the perfect hexagonal honeycomb is identical to that of the other two lattices but the shear modulus G is dominated by bending and scales with $\bar{\rho}^3$.

3 Prediction of the Elastic Moduli of Imperfect Lattices

The finite element method was used to calculate the macroscopic modulus of imperfect lattices containing either missing bars or a dispersion of nodes. The effective elastic moduli were the mean values from multiple realizations of the imperfect lattices. The response of each realization has been determined by the finite element computer package ABAQUS/STANDARD [22]. In the simulations, a two-noded cubic beam element was used to represent each wall (element "B23" within ABAQUS). The B23 element is an Euler–Bernoulli beam that can both stretch and bend, but is rigid against shear.

A preliminary set of finite element calculations was performed to explore the degree to which a neglect of shear deformation leads to inaccurate predictions of the macroscopic stiffness. A perfect, hexagonal honeycomb was analyzed, and the macroscopic shear modulus was calculated using the B23 elements within ABAQUS. The results were then compared to the analytic expression given by Silva et al. [7], which includes the contribution from shear of the beam cross section. A comparison reveals that the neglect of shear deformation leads to too stiff a macroscopic shear modulus by a negligible factor: The finite element prediction is too stiff by 2.6% at a relative density of $\bar{\rho}=0.115$, with a smaller error at lower values of relative density. The Kagome and triangulated lattices are stretching-controlled structures and so the error introduced by a neglect of shear deformation is more minor for these lattices than the hexagonal honeycomb. We conclude that the B23 elements within ABAQUS are adequate for our purposes.

The macroscopic stiffness of the imperfect lattices was obtained by considering a representative unit cell containing a random distribution of imperfections. Periodic boundary conditions were applied such that the translation displacements u_α^i and rotation θ^i of every node on the boundary of the mesh satisfy

$$u_\alpha^J - u_\alpha^I = \varepsilon_{\alpha\beta}(x_\beta^J - x_\beta^I), \quad \theta^J - \theta^I = 0, \quad \alpha,\beta = 1,2 \tag{1}$$

where $\varepsilon_{\alpha\beta}$ is the average macroscopic strain and x_β^J and x_β^I are the coordinates of a pair of corresponding nodes I and J on opposite sides of the mesh. Periodic boundary conditions were also adopted in numerical studies of imperfection sensitivity of cellular materials by Chen [11,23], Grenestedt [10], Zhu et al. [8], and Gan et al. [12]. Li et al. [17] confirmed that for a regular honeycomb the computed moduli are independent of the number of cells with periodic boundary conditions; this is not the case for an analysis based on displacement boundary conditions.

In each simulation, a single-step linear calculation was performed to determine the response to uniform biaxial (hydrostatic) strain and then to deviatoric strain. The bulk modulus and shear modulus were calculated from the work-conjugate applied loads. Meshes of the imperfect lattice were generated from perfect parent meshes using a MATLAB [24] routine. For the imperfection of missing bars, the routine randomly removed a proportion f of the elements (see Fig. 2). For misplaced nodes, the routine displaced every node in the mesh from its perfect position along a randomly generated direction by a randomly generated distance up to a maximum distance of al, where a is the amplitude of nodal dispersion. The probability distribution of the random radial movement of nodes was chosen to give a uniform probability distribution with respect to area within a disk of radius al. Corresponding pairs of boundary nodes were assigned the same random displacement to ensure periodicity. Note that a can take values of the range of 0 to 0.5; for the choice $a=0.5$, occasional nodes touch in the undeformed configuration.

The unit cell of the imperfect lattice in the finite element simulations should be sufficiently large in order to give an accurate estimate of the modulus for the infinite imperfect lattice. A series of preliminary calculations was performed with the side length L of the unit cell varied from $12l$ to $96l$. For a given size of unit cell, 20 different randomly generated structural realizations were analyzed with the same level of imperfection in the form of missing bars. With increasing size of unit cell, it is anticipated that the standard deviation of modulus decreases (and asymptotes to zero in the limit of an infinite mesh), while the mean value of modulus asymptotes to the infinite lattice result. It is of interest to explore whether the rate of convergence of the standard deviation is the

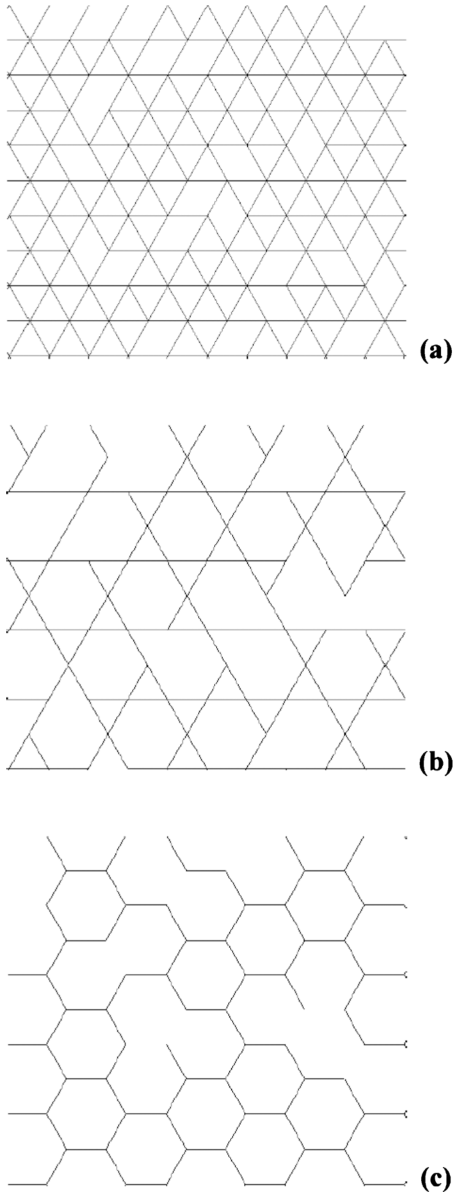

Fig. 2 Planar grids with f=0.1 (10%) missing bars: (a) triangular, (b) Kagome, and (c) hexagonal

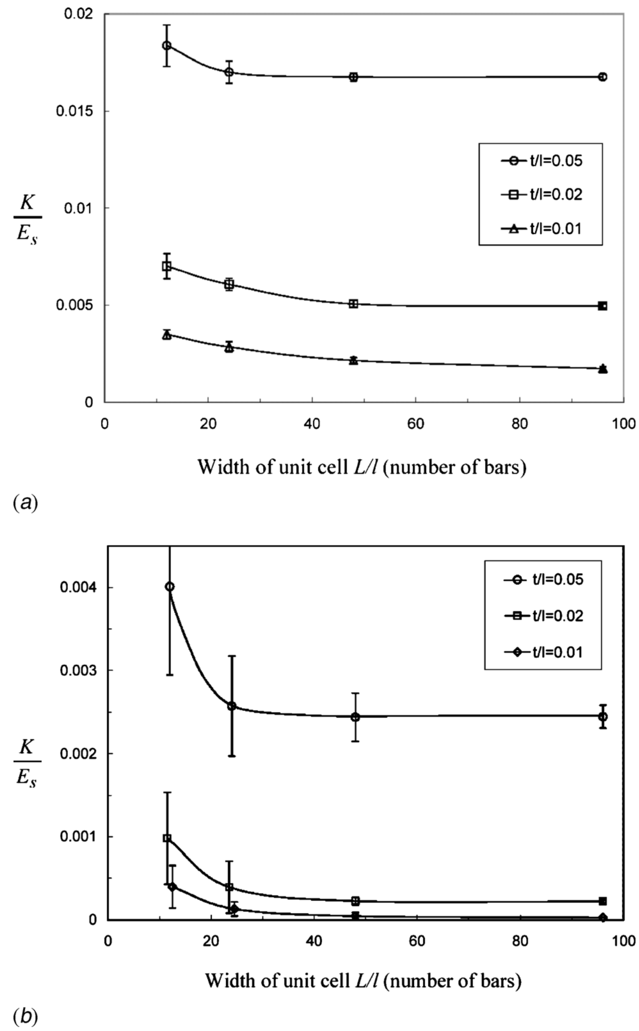

Fig. 3 Convergence of bulk modulus of planar Kagome with increasing size of unit cell; (a) f=0.01 (1%) missing bars and (b) f=0.1 (10%) missing bars (means of 20 simulations plotted, error bars represent ±1 standard deviation)

same as that for the mean value. A similar approach was adopted by Gan et al. [12]. They examined the mean and standard deviation of the moduli of randomly perturbed (Voronoi) 3D open cell foams and showed convergence of the results for increasing model size.

A representative set of calculations is reported in Fig. 3, for a Kagome unit cell with selected values of side lengths 12l, 24l, 48l and 96l. In each case, the unit cell was rectangular with a height to width ratio of $\sqrt{3}/2$. The mean and standard deviation of the bulk modulus K were obtained from each set of 20 simulations. This exercise was performed for three different values of t/l (i.e., varying $\bar{\rho}$) and for two fractions of missing bars $f=0.01$ and $f=0.1$. It is clear from the results shown in Fig. 3 that the standard deviation of the bulk modulus K asymptotes to zero with increasing size of unit cell. The mean value decreases to a stabilized value, and this stable value is taken to represent the infinite lattice modulus. We note that the required size of unit cell to give a negligible standard deviation is essentially the same as that required to give a stable mean value. For any given size of unit cell and relative density, the standard deviation of the bulk modulus is lower for a smaller level of imperfection f of missing bars.

The Kagome lattice has the characteristic feature that the size of unit cell in order to give an acceptable estimate for the infinite lattice solution is dependent on t/l. For example, for the choice $f=0.01$, it is evident from Fig. 3(a) that a unit cell of width $L=24l$ is adequate for $t/l=0.05$, whereas a unit cell of width $L=96l$ is needed for $t/l=0.01$. The reason for this size dependence on t/l is evident from an examination of the deformed mesh, as follows. Consider the choice of a unit cell of size $L=96l$ and $f=0.01$. Then, the perturbation in deformation field around a missing bar is much larger for $t/l=0.02$ than for $t/l=0.05$, as shown in Fig. 4. This has already been noted in the study of Wicks and Guest [25] in the context of single member actuation of a Kagome lattice.

Similar preliminary investigations have been performed for the shear modulus of the Kagome lattice, and for both in-plane moduli of the triangular lattice and hexagonal honeycomb. The required sizes of unit cell to achieve accurate values of the effective shear modulus of the Kagome lattice are the same as that noted above for the bulk modulus. Smaller unit cells suffice for the triangular lattice and hexagonal honeycomb as these lattices do not have long decay lengths adjacent to missing bars, as dis-

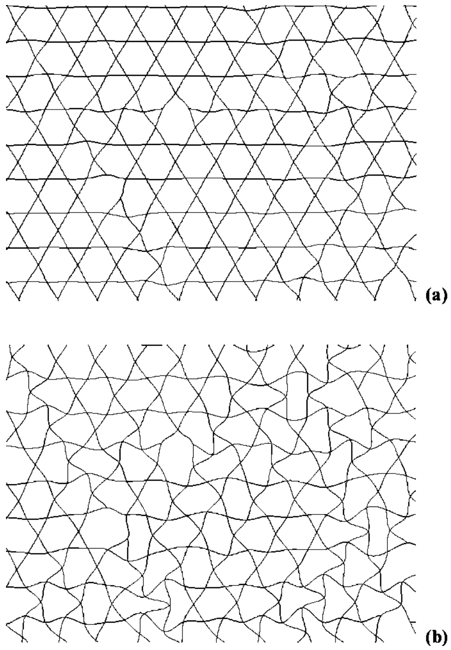

Fig. 4 Planar Kagome grids with f=0.01 (1%) missing bars: (*a*) equibiaxial strain t/l=0.05; (*b*) equibiaxial strain t/l=0.02

cussed by Wicks and Guest [25]. Detailed results for the effective moduli are now given for the three lattices employing a unit cell of width 96l.

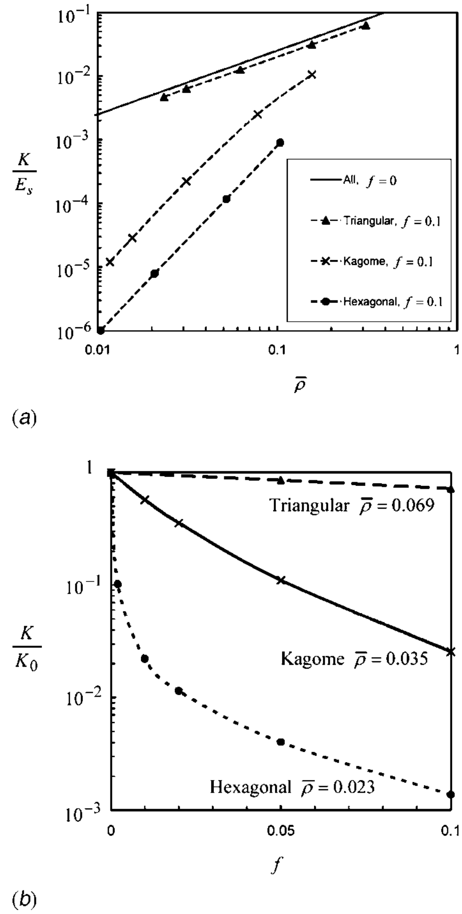

Fig. 5 Sensitivity of bulk modulus of planar grids to missing bars: (*a*) for varying relative density $\bar{\rho}$ with f=0 and f=0.1 (10%) missing bars; (*b*) for varying proportion of missing bars f with fixed cell-wall thickness t/l=0.02

4 Imperfection Sensitivity

4.1 Imperfection Type 1: Missing Bars. The first imperfection investigated is that of missing (or broken) bars. All nodes are in their ideal (perfect) positions and all bars are perfectly straight, but a random distribution of the bars are missing. Figure 2 shows representative regions of the three planar lattices where the fraction of missing bars f=0.1.

4.1.1 Bulk Modulus. The dependence of bulk modulus of the three lattices on relative density $\bar{\rho}$ is plotted in Fig. 5(*a*) for f=0 and f=0.1. (Note that the relative density as plotted includes the correction associated with missing bars.) All lattices in their perfect state (f=0) have a slope of unity on this log-log plot. This is due to the stretching-dominated behavior: the bulk modulus scales linearly with $\bar{\rho}$.

The triangulated lattice is relatively unaffected by the introduction of missing bars; the modulus drops by 30% upon increasing f from zero to 0.1. However, the bulk modulus still scales linearly with $\bar{\rho}$ (a slope of unity), indicating that stretching behavior is preserved.

In contrast, for the Kagome and hexagonal lattices, an increase of f from 0 to 0.1 greatly reduces the bulk modulus. The bulk modulus now scales as $\bar{\rho}^3$ indicating that bending behavior now dominates. However, for the practical range of relative density, the Kagome is one order of magnitude stiffer than the honeycomb and is therefore much more tolerant to imperfections in the form of missing bars.

The knockdown in bulk modulus K/K_0 is plotted as a function of fraction f of missing bars in Fig. 5(*b*) for the three lattices. In this plot, an intermediate cell-wall (bar) thickness has been chosen of t/l=0.02 and K_0 is the bulk modulus of each lattice for f=0. (Note that a fixed value of t/l gives a different $\bar{\rho}$ for each lattice.) The extreme imperfection sensitivity of the hexagonal lattice is evident. In contrast the Kagome is much more imperfection tolerant and the triangular lattice is almost insensitive to missing bars.

4.1.2 Shear Modulus. The effective shear modulus G of the three lattices is plotted as a function of $\bar{\rho}$ in Fig. 6(*a*) for the two choices f=0 and f=0.1. For the perfect lattice, f=0, the triangulated and Kagome lattices have the same stiffness, as expected from Table 1. However, for f=0.1, the Kagome suffers a significant knockdown in shear stiffness whereas the triangulated grid is relatively unaffected by the imperfection. The perfect hexagonal honeycomb has a very low shear modulus due to its bending-dominated response, and the introduction of missing bars leads to only a small additional reduction in modulus.

The knockdown in shear modulus G/G_0 with increasing f is plotted in Fig. 6(*b*) for the choice t/l=0.02, where G_0 is the bulk modulus of each lattice for f=0. The Kagome is clearly the most sensitive to the introduction of missing bars but it must be recalled from Fig. 6(*a*) that the absolute shear modulus of the Kagome lattice is an order of magnitude greater than that of the hexagonal honeycomb at any given value of $\bar{\rho}$.

4.2 Imperfection Type 2: Stochastic Dispersion of Nodes. The second type of imperfection investigated is that of misplaced nodes. All bars are present and perfectly straight but are connected to nodes which are randomly displaced from their ideal (perfect) positions. As already noted, the random displacement has a uniform probability density function over a circular disk of radius al. Figure 7 shows the representative portions of each lattice with stochastically displaced nodes of amplitude a=0.3.

4.2.1 Bulk Modulus. The dependence of in-plane bulk modulus on relative density for the three lattices is shown in Fig. 8(*a*),

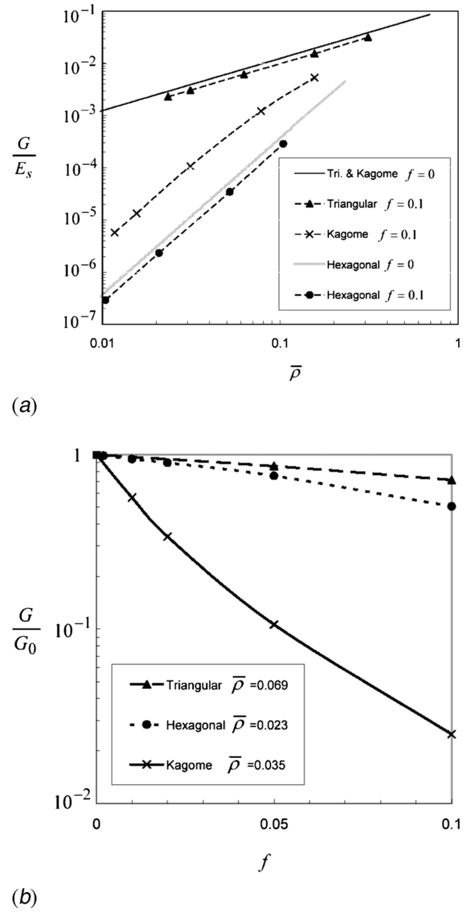

Fig. 6 Sensitivity of shear modulus of planar grids to missing bars: (*a*) for varying relative density with $f=0$ and $f=0.1$ missing bars; (*b*) for cell-wall thickness $t/l=0.02$

for the choices $a=0$ and $a=0.5$. The triangulated lattice is relatively insensitive to nodal position. The bulk modulus of the imperfect triangulated lattice linearly scales with $\bar{\rho}$, indicating that stretching behavior remains dominant. In contrast, the stiffness of the hexagonal lattice is severely degraded by random movement of the nodes. The bulk modulus K of the imperfect honeycomb scales as $\bar{\rho}^3$. The Kagome is the intermediate case: The bulk modulus of the imperfect lattice scales as $\bar{\rho}^n$, with the index n increasing from unity to 2 with increasing a. In the extreme case shown for $a=0.5$, K scales as $\bar{\rho}^2$. It is argued that this intermediate behavior of the Kagome lattice is due to the fact that it deforms by a combination of bending and stretching, as discussed by Wicks and Guest [25]. The knockdown in bulk modulus K/K_0 is plotted as a function of dispersion amplitude a in Fig. 8(*b*) for the case $t/l=0.02$. This plot clearly shows the extreme sensitivity of the hexagonal honeycomb to the imperfection, while the Kagome lattice is moderately sensitive and the triangular lattice is almost insensitive.

4.2.2 Shear Modulus. Figures 9(*a*) and 9(*b*) show the effect of stochastic nodal dispersion on the shear modulus of the three lattices, again for $a=0$ and $a=0.5$. The shear modulus of the triangulated lattice decreases slightly while that of the hexagonal honeycomb increases slightly when a is increased from zero (perfect lattice) to 0.5 (imperfect lattice). In contrast, the shear modulus of the Kagome lattice is sensitive to nodal dispersion, as shown in Fig. 9(*b*). For the choice $a=0.5$, as shown in Fig. 9(*a*), the shear modulus approximately scales as $\bar{\rho}^2$. A similar behavior has already been noted for the bulk modulus.

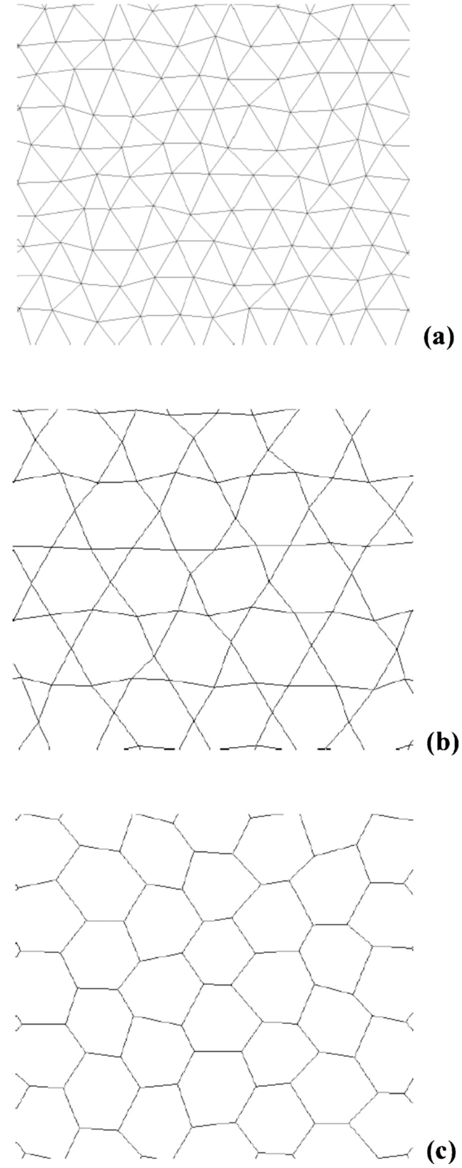

Fig. 7 Planar grids with stochastic nodal dispersion amplitude $a=0.3$: (*a*) triangular, (*b*) Kagome, and (*c*) hexagonal

The slight effect of nodal dispersion on the shear modulus of the hexagonal honeycomb is consistent with the observations by Silva et al. [7] that the shear modulus of a Voronoi honeycomb (with a nodal connectivity of $Z=3$) is slightly above that of the perfect hexagonal honeycomb.

4.3 Imperfection Type 3: Bar Waviness. The final type of imperfection considered here is bar waviness, that is, lack of straightness of cell walls. The waviness of a bar leads to a reduction in axial stiffness and to a negligible change of bending stiffness. The relationship between amplitude of waviness and axial stiffness of a bar can be straightforwardly determined, as follows.

Consider a bar of length l, with an integral number of sine waves of wavelength l'. Then, the misalignment w as a function of position along the bar x is

$$w(x) = w_0 \sin\left(\frac{2\pi x}{l'}\right) \quad (2)$$

where w_0 is the amplitude of waviness, as defined in Fig. 10(*a*). Now apply an axial tension T to the ends of the bar. This tension

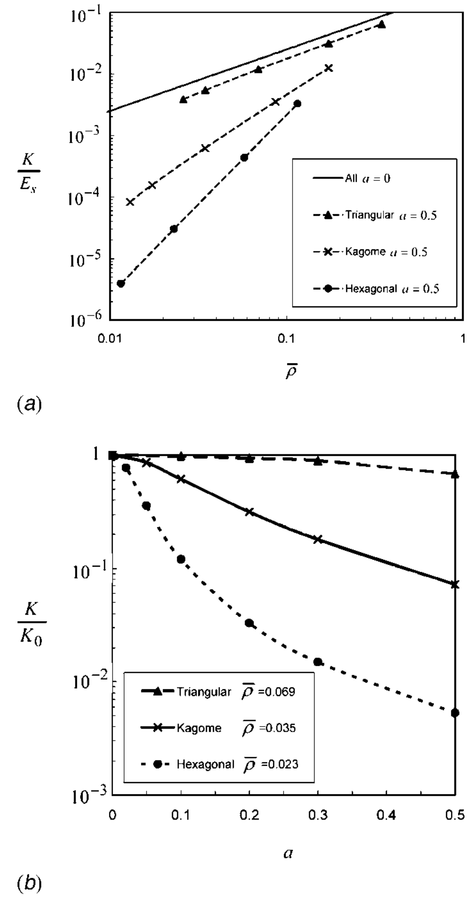

Fig. 8 Sensitivity of bulk modulus of planar grids to stochastic nodal dispersion: (*a*) for varying relative density with dispersion amplitude $a=0$ and $a=0.5$; (*b*) for varying dispersion amplitude with fixed cell-wall thickness $t/l=0.02$

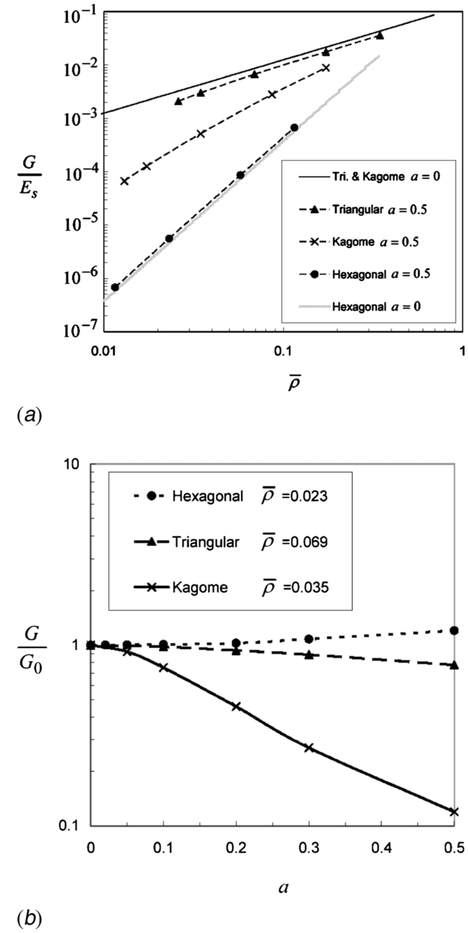

Fig. 9 Sensitivity of shear modulus of planar grids to stochastic nodal dispersion: (*a*) for varying relative density with dispersion amplitude $a=0$ and $a=0.5$; (*b*) for varying dispersion amplitude with fixed cell-wall thickness $t/l=0.02$

gives rise to a bending moment M and an additional transverse displacement u such that

$$M = -EI\frac{d^2u}{dx^2} = Tw(x) \tag{3}$$

where E is the axial modulus and I the second moment of area of the bar cross section.

$$I = t^3/12 \tag{4}$$

Substitute Eq. (2) into Eq. (3) and solve the resulting differential equation to obtain

$$u(x) = -\frac{T}{EI}\left(\frac{l'}{2\pi}\right)^2 w_0 \sin\left(\frac{2\pi x}{l'}\right) \tag{5}$$

The extension Δl of a bar of thickness t and total length l has two contributions: the stretching of the bar and the straightening due to its initial waviness. Hence,

$$\Delta l = \frac{Tl}{Et} - \int_0^l \left[\sqrt{1 + \left(\frac{du}{dx} + \frac{dw}{dx}\right)^2} - \sqrt{1 + \left(\frac{dw}{dx}\right)^2}\right] dx \tag{6}$$

Upon assuming that the elastic deflection $du/dx \ll dw/dx \ll 1$, we obtain

$$\Delta l = \frac{Tl}{Et}\left(1 + \frac{w_0^2 t}{2I}\right) \tag{7}$$

The axial stiffness k of the bar is therefore

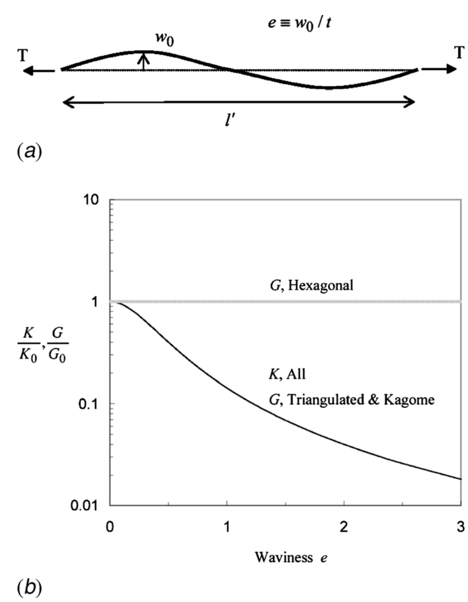

Fig. 10 (*a*) Definition of bar waviness; (*b*) sensitivity of bulk and shear moduli of triangular, Kagome, and hexagonal planar grids to bar waviness

$$k = \frac{T}{\Delta l} = \frac{Et}{l} \frac{1}{(1 + 6e^2)} \quad (8)$$

where $e \equiv w_0 / t$ is a nondimensional measure of waviness given by the ratio of the amplitude of waviness w_0 to the bar thickness t. This result agrees with the knockdown factor calculated by Grenestedt [14].

Thus, the presence of waviness reduces the axial stiffness of the bar by the factor $1+6e^2$. Consequently, stretching-dominated deformation modes of lattices will have the relevant modulus reduced by this factor upon introducing bar waviness. This knockdown factor is potent: The effective modulus drops by a factor of 25 for a waviness amplitude $w_0=2t$. For the three isotropic lattices considered in this study, all moduli are degraded except for the shear modulus of the hexagonal honeycomb. This result is shown in Fig. 10(*b*).

5 Concluding Remarks

Perfect triangulated and Kagome lattices are stiff, stretching-dominated structures under all loadings and their bulk and shear moduli scale with $\bar{\rho}$. The hexagonal honeycomb is only stretching dominated under hydrostatic loading. Under deviatoric loading, it is a flexible, bending-dominated structure. Consequently, the bulk modulus of the perfect hexagonal honeycomb scales with $\bar{\rho}$, whereas its shear modulus scales with $\bar{\rho}^3$.

The high nodal connectivity $Z=6$ of the triangulated lattice confers insensitivity to imperfections in the form of missing bars and nodal dispersion: It remains a stiff stretching structure under all loadings. In contrast, the bulk and shear moduli of the Kagome lattice ($Z=4$) are significantly degraded by these imperfections. The bulk modulus of the hexagonal lattice ($Z=3$) is extremely sensitive to the presence of missing bars and nodal dispersion as the deformation mode switches from stretching dominated to bending dominated. Its shear modulus is almost unaffected since it is bending governed for both the perfect and imperfect geometries.

This study has highlighted the relative ranking of the in-plane moduli of three isotropic lattices in the presence of missing bars and nodal dispersion. For the same relative density and level of imperfection, both the bulk and shear moduli of the Kagome are intermediate between those of the imperfection-insensitive triangulated lattice and the hexagonal honeycomb. This suggests that the Kagome lattice is a promising topology for morphing applications: The imperfect structure is stiff under external loads, yet compliant when one or more bars are axially actuated.

Cell wall lack of straightness (bar waviness) degrades the moduli of all three lattices where the behavior is stretching dominated. Consequently, it is only the shear modulus of the hexagonal honeycomb that is insensitive to this imperfection, since in this case the behavior is already bending controlled.

References

[1] Deshpande, V. S., Ashby, M. F., and Fleck, N. A., 2001, "FoamTopology Bending Versus Stretching Dominated Architectures," Acta Mater., **49**, pp. 1035–1040.

[2] Maxwell, J. C., 1864, "On the Calculation of the Equilibrium and Stiffness of Frames," Philos. Mag., **27**, pp. 294–299, 1890, *Collected Papers, XXVI*, Cambridge University Press, Cambridge.

[3] Hyun, S., and Torquato, S., 2002, "Optimal and Manufacturable Two-Dimensional, Kagomé-Like Cellular Solids," J. Mater. Res., **17**(1), pp. 137–144.

[4] Hutchinson, R. G., and Fleck, N. A., 2006, "The Structural Performance of the Periodic Truss," J. Mech. Phys. Solids, **54**(4), 756–782.

[5] Symons, D. D., Hutchinson, R. G., and Fleck, N. A., 2005, "Actuation of the Kagome Double Layer Grid. Part 1: Prediction of Performance of the Perfect Structure," J. Mech. Phys. Solids, **53**, pp. 1855–1874.

[6] Fleck, N. A., and Qiu, X., 2006, "The Damage Tolerance of Elastic-Brittle, Two-Dimensional Isotropic Lattices," J. Mech. Phys. Solids, **55**, 562–588.

[7] Silva, M. J., Hayes, W. C., and Gibson, L. J., 1995, "The Effects of Non-Periodic Microstructure on the Elastic Properties of Two-Dimensional Cellular Solids," Int. J. Mech. Sci., **37**(11), 1161–1177.

[8] Zhu, H. X., Hobdell, J. R., and Windle, A. H., 2001, "Effects of Cell Irregularity on the Elastic Properties of 2D Voronoi Honeycombs," J. Mech. Phys. Solids, **49**, 857–870.

[9] Van Der Burg, M. W. D., and Shulmeister, V., Van Der Geissen, E., Marissen, R., 1997, "On the Linear Elastic Properties of Regular and Random Open-Cell Foam Models," J. Cell. Plast., **33**(1), 31–54.

[10] Grenestedt, J. L., and Tanaka, K., 1999, "Influence of Cell Shape Variations on Elastic Stiffness of Closed Cell Cellular Solids," Scr. Mater., **40**(1), 71–77.

[11] Chen, C., Lu, T. J., and Fleck, N. A., 2001, "Effect of Inclusions and Holes on the Stiffness and Strength of Honeycombs," Int. J. Mech. Sci., **43**, pp. 487–504.

[12] Gan, Y. X., Chen, C., and Shen, Y. P., 2005, "Three-Dimensional Modeling of the Mechanical Property of Linearly Elastic Open Cell Foams," Int. J. Solids Struct., **42**, pp. 6628–6642.

[13] Wallach, J. C., and Gibson, L. J., 2001, "Defect Sensitivity of a 3D Truss Material," Scr. Mater., **45**(6), pp. 639–644.

[14] Grenestedt, J. L., 1998, "Influence of Wavy Imperfections in Cell Walls on Elastic Stiffness of Cellular Solids," J. Mech. Phys. Solids, **46**(1), pp. 29–50.

[15] Simone, A. E., and Gibson, L. J., 1998, "The Effects of Cell Face Curvature and Corrugations on the Stiffness and Strength of Metallic Foams," Acta Mater., **46**(11), pp. 3929–3935.

[16] Grenestedt, J. L., 2005, "On Interactions Between Imperfections in Cellular Solids," J. Mater. Sci., **40**, pp. 5853–5857.

[17] Li, K., Gao, X.-L., and Subhash, G., 2005, "Effects of Cell Shape and Cell Wall Thickness Variations on the Elastic Properties of Two-Dimensional Cellular Solids," Int. J. Solids Struct., **42**, pp. 1777–1795.

[18] Symons, D. D., Shieh, J., and Fleck, N. A., 2005, "Actuation of the Kagome Double Layer Grid. Part 2: Effect of Imperfections on the Measured and Predicted Actuation Stiffness," J. Mech. Phys. Solids, **53**, pp. 1875–1891.

[19] Gibson, L. J., and Ashby, M. F., 1997, *Cellular Solids: Structure and Properties*, 2nd ed., Cambridge University Press, Cambridge.

[20] Christensen, R. M., 2000, "Mechanics of Cellular and Other Low-Density Materials," Int. J. Solids Struct., **37**, pp. 93–104.

[21] Srikantha Phani, A., Woodhouse, J., and Fleck, N. A., 2006, "Wave Propagation in Two-Dimensional Periodic Lattices," J. Acoust. Soc. Am., **119**(4), pp. 1995–2005.

[22] HKS, 2003, ABAQUS/STANDARD, Version 6.4.1, Hibbit, Karlsson and Sorenson Inc., Providence, RI.

[23] Chen, C., Lu, T. J., and Fleck, N. A., 1999, "Effect of Imperfections on the Yielding of Two-Dimensional Foams," J. Mech. Phys. Solids, **47**, pp. 2235–2272.

[24] MathWorks, 2002, MATLAB, Version 6.5, The MathWorks Inc., Natick, MA.

[25] Wicks, N., and Guest, S. D., 2004, "Single Member Actuation in Large Repetitive Truss Structures," Int. J. Solids Struct., **41**, pp. 965–978.